Relativistic wavefunctions on the Poincaré group

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Abstract
The Biedenharn type relativistic wavefunctions are considered on the group
manifold of the Poincaré group. It is shown that the wavefunctions can be
factorized on the group manifold into translation group and Lorentz group
parts. A Lagrangian formalism and field equations for such factorizations
are given. Parametrizations of the functions obtained are studied in terms of
a ten-parameter set of the Poincaré group. An explicit construction of the
wavefunction for spin $1/2$ is given. A relation of the proposed description with
quantum field theory and harmonic analysis on the Poincaré group is discussed.

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1. Introduction

In 1988, Biedenharn *et al* [5] introduced the Poincaré group representations of wavefunctions
on the space of complex spinors. The construction presented in [5] is an extension of the
Wigner group theoretical method [37]. On the other hand, in accordance with basic principles
of quantum field theory, a wavefunction of the particle is a solution of some relativistic wave
equation. Relativistic wavefunctions in the work [5] were introduced without explicit reference
to wave equations and for this reason they represent purely group theoretical constructions.
However, if we further develop the Poincaré group representations of wavefunctions with
reference to such basic notions of QFT as a Lagrangian and wave equations, then we come to
a quantum field theory on the Poincaré group[1] (QFTPG) introduced by Lurçat in 1964 [20]
(see also [3, 18, 7, 2, 19, 31, 21, 8, 11, 12] and references therein). In contrast to the standard
QFT (QFT in the Minkowski spacetime) case, for QFTPG all the notions and quantities are
constructed on a ten-dimensional group manifold $\mathfrak{F}$ of the Poincaré group. It should be noted
here that a construction of relativistic wave equation theory on the group manifold $\mathfrak{F}$ is one

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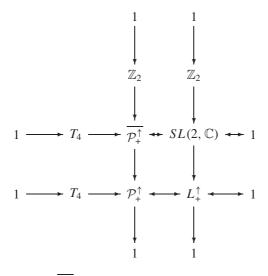[1] In 1955, Finkelstein [9] showed that elementary particle models with internal degrees of freedom can be described
on manifolds larger then Minkowski spacetime (homogeneous spaces of the Poincaré group). Consideration of the
field models on the homogeneous spaces leads to a generalization of the concept of wavefunction. One of the first
examples of such generalized wavefunctions was studied by Nilsson and Beskow [23].

of the primary problems in this area, which remains incompletely solved. Wavefunctions and wave equations on a six-dimensional submanifold $\mathfrak{L} \subset \mathfrak{F}$, which is a group manifold of the Lorentz group, have been studied in the recent work [33].

In the present paper we consider Biedenharn type relativistic wavefunctions on the group manifold $\mathfrak{F}$. It is shown that the general form of the wavefunctions inherits its structure from the semidirect product $SL(2, \mathbb{C}) \odot T_4$ and for that reason the wavefunctions on $\mathfrak{F}$ are represented by a factorization $\psi(x)\psi(\mathfrak{g})$, where $x \in T_4$, $\mathfrak{g} \in SL(2, \mathbb{C})$. Using a Lagrangian formalism on the tangent bundle $T\mathfrak{F}$ of the manifold $\mathfrak{F}$, we obtain field equations separately for the parts $\psi(x)$ and $\psi(\mathfrak{g})$. Solutions of the field equations for $\psi(x)$ can be obtained via the usual plane-wave approximation. In turn, solutions of the field equations with $\psi(\mathfrak{g})$ have been found in the form of expansions in associated hyperspherical functions[2]. The wavefunction on the Poincaré group in the case of the $(1/2, 0) \oplus (0, 1/2)$ representation space, usually related to an electron–positron field, is considered by way of an example.

2. Preliminaries

Let us consider some basic facts concerning the Poincaré group $\mathcal{P}$. First of all, the group $\mathcal{P}$ has the same number of connected components as the Lorentz group. Later on we will consider only the component $\mathcal{P}_+^\uparrow$ corresponding the connected component $L_+^\uparrow$ (the so-called special Lorentz group [26]). As is known, a universal covering $\overline{\mathcal{P}_+^\uparrow}$ of the group $\mathcal{P}_+^\uparrow$ is defined by a semidirect product $\overline{\mathcal{P}_+^\uparrow} = SL(2, \mathbb{C}) \odot T_4 \simeq \mathbf{Spin}_+(1, 3) \odot T_4$, where T_4 is a subgroup of four-dimensional translations. The relations between the groups $\overline{\mathcal{P}_+^\uparrow}$, $\mathcal{P}_+^\uparrow$, $SL(2, \mathbb{C})$ and $L_+^\uparrow$ are defined by the following diagram of exact sequences:

$$
\begin{array}{ccc}
1 & & 1 \\
\downarrow & & \downarrow \\
\mathbb{Z}_2 & & \mathbb{Z}_2 \\
\downarrow & & \downarrow \\
1 \longrightarrow T_4 \longrightarrow \overline{\mathcal{P}_+^\uparrow} \twoheadleftarrow SL(2, \mathbb{C}) \twoheadleftarrow 1 \\
\downarrow \quad\quad\quad \downarrow \\
1 \longrightarrow T_4 \longrightarrow \mathcal{P}_+^\uparrow \longleftrightarrow L_+^\uparrow \longleftrightarrow 1 \\
\downarrow \quad\quad\quad \downarrow \\
1 \quad\quad\quad 1
\end{array}
$$

The diagram shows that $\overline{\mathcal{P}_+^\uparrow}$ $\left(\mathcal{P}_+^\uparrow\right)$ is the semidirect product of $SL(2, \mathbb{C})$ $\left(L_+^\uparrow\right)$ and T_4.

Then each transformation $T_\alpha \in \mathcal{P}_+^\uparrow$ is defined by a parameter set $\alpha(\alpha_1, \ldots, \alpha_{10})$, which can be represented by a point of the space $\mathfrak{F}_{10}$. The space $\mathfrak{F}_{10}$ possesses locally Euclidean properties; therefore, it is a manifold, called *a group manifold of the Poincaré group*. It is easy to see that the set α can be divided into two subsets, $\alpha(x_1, x_2, x_3, x_4, \mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{g}_3, \mathfrak{g}_4, \mathfrak{g}_5, \mathfrak{g}_6)$,

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[2] Matrix elements of both spinor and principal series representations of the Lorentz group are expressed via the hyperspherical functions [32, 34].

where $x_i \in T_4$ are parameters of the translation subgroup, $\mathfrak{g}_j$ are parameters of the group $SL(2, \mathbb{C})$. In turn, the transformation $T_{\mathfrak{g}}$ is defined by a set $\mathfrak{g}(\mathfrak{g}_1, \dots, \mathfrak{g}_6)$, which can be represented by a point of a six-dimensional submanifold $\mathfrak{L}_6 \subset \mathfrak{F}_{10}$, called *a group manifold of the Lorentz group*.

In the present paper we restrict ourselves to consideration of finite dimensional representations of the Poincaré group. The group T_4 of four-dimensional translations is an Abelian group, formed as a direct product of the four one-dimensional translation groups, each of which is isomorphic to an additive group of real numbers. Hence it follows that all irreducible representations of T_4 are one dimensional and expressed via the exponential. In turn, as shown by Naimark [22], spinor representations exhaust all the finite dimensional irreducible representations of the group $SL(2, \mathbb{C})$. Any spinor representation of $SL(2, \mathbb{C})$ can be defined in the space of symmetric polynomials in the following form:

$$p(z_0, z_1, \bar{z}_0, \bar{z}_1) = \sum_{\substack{(\alpha_1, \dots, \alpha_k) \\ (\dot{\alpha}_1, \dots, \dot{\alpha}_r)}} \frac{1}{k!r!} a^{\alpha_1 \cdots \alpha_k \dot{\alpha}_1 \cdots \dot{\alpha}_r} z_{\alpha_1} \cdots z_{\alpha_k} \bar{z}_{\dot{\alpha}_1} \cdots \bar{z}_{\dot{\alpha}_r} \qquad (\alpha_i, \dot{\alpha}_i = 0, 1) \qquad (1)$$

where the numbers $a^{\alpha_1 \cdots \alpha_k \dot{\alpha}_1 \cdots \dot{\alpha}_r}$ are unaffected at the permutations of indices. The expressions (1) can be understood as *functions on the Lorentz group*. When the coefficients $a^{\alpha_1 \cdots \alpha_k \dot{\alpha}_1 \cdots \dot{\alpha}_r}$ in (1) depend on the variables $x_i \in T_4$ ($i = 1, 2, 3, 4$), we come to the Biedenharn type functions [5]:

$$p(x, z, \bar{z}) = \sum_{\substack{(\alpha_1, \dots, \alpha_k) \\ (\dot{\alpha}_1, \dots, \dot{\alpha}_r)}} \frac{1}{k!r!} a^{\alpha_1 \cdots \alpha_k \dot{\alpha}_1 \cdots \dot{\alpha}_r}(x) z_{\alpha_1} \cdots z_{\alpha_k} \bar{z}_{\dot{\alpha}_1} \cdots \bar{z}_{\dot{\alpha}_r}. \qquad (\alpha_i, \dot{\alpha}_i = 0, 1) \qquad (2)$$

The functions (2) should be considered as *the functions on the Poincaré group*[3]. Some applications of these functions are contained in [35, 11]. Representations of the Poincaré group $SL(2, \mathbb{C}) \odot T(4)$ are realized via the functions (2).

3. Field equations on the Poincaré group

Let $\mathcal{L}(\alpha)$ be a Lagrangian on the group manifold $\mathfrak{F}$ of the Poincaré group (in other words, $\mathcal{L}(\alpha)$ is a ten-dimensional point function), where α is the parameter set of this group. Then we will call an integral for $\mathcal{L}(\alpha)$ on some ten-dimensional volume Ω of the group manifold *an action on the Poincaré group*:

$$A = \int_{\Omega} d\alpha \, \mathcal{L}(\alpha)$$

where $d\alpha$ is a Haar measure on the group $\mathcal{P}$.

Let $\psi(\alpha)$ be a function on the group manifold $\mathfrak{F}$ (now it is sufficient to assume that $\psi(\alpha)$ is a square integrable function on the Poincaré group) and let

$$\frac{\partial \mathcal{L}}{\partial \psi} - \frac{\partial}{\partial \alpha} \frac{\partial \mathcal{L}}{\partial \frac{\partial \psi}{\partial \alpha}} = 0 \qquad (3)$$

be Euler–Lagrange equations on $\mathfrak{F}$ (more precisely speaking, the equations (3) act on the tangent bundle $T\mathfrak{F} = \cup_{\alpha \in \mathfrak{F}} T_\alpha \mathfrak{F}$ of the manifold $\mathfrak{F}$; see [1]). Let us introduce a Lagrangian $\mathcal{L}(\alpha)$ depending on the field function $\psi(\alpha)$ as follows:

$$\mathcal{L}(\alpha) = -\frac{1}{2} \left(\psi^*(\alpha) B_\mu \frac{\partial \psi(\alpha)}{\partial \alpha_\mu} - \frac{\partial \psi^*(\alpha)}{\partial \alpha_\mu} B_\mu \psi(\alpha) \right) - \kappa \psi^*(\alpha) B_{11} \psi(\alpha)$$

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[3] Antisymmetrizing the variables z and $\bar{z}$ in (2), we come to the functions on the supergroup. In the particular case of Grassmann variables [4] we have the functions on a so-called super-Poincaré group.

where B_ν ($\nu = 1, 2, \ldots, 10$) are square matrices. The number of rows and columns in these matrices is equal to the number of components of $\psi(\alpha)$; κ is a non-null real constant.

Further, if B_{11} is non-singular, then we can introduce the matrices

$$\Gamma_\mu = B_{11}^{-1} B_\mu \qquad \mu = 1, 2, \ldots, 10$$

and represent the Lagrangian $\mathcal{L}(\alpha)$ in the form

$$\mathcal{L}(\alpha) = -\frac{1}{2}\left(\overline{\psi}(\alpha)\Gamma_\mu \frac{\partial\psi(\alpha)}{\partial\alpha_\mu} - \frac{\overline{\psi}(\alpha)}{\partial\alpha_\mu}\Gamma_\mu\psi(\alpha)\right) - \kappa\overline{\psi}(\alpha)\psi(\alpha) \qquad (4)$$

where

$$\overline{\psi}(\alpha) = \psi^*(\alpha)B_{11}.$$

As a direct consequence of (2), the relativistic wavefunction $\psi(\alpha)$ on the group manifold $\mathfrak{F}$ is represented by the following factorization:

$$\psi(\alpha) = \psi(x)\psi(\mathfrak{g}) = \psi(x_1, x_2, x_3, x_4)\psi(\varphi, \epsilon, \theta, \tau, \phi, \varepsilon) \qquad (5)$$

where $\psi(x_i)$ is a function depending on the parameters of the subgroup T_4, $x_i \in T_4$ ($i = 1, \ldots 4$), and $\psi(\mathfrak{g})$ is a function on the Lorentz group, where the six parameters of this group are defined by the Euler angles $\varphi, \epsilon, \theta, \tau, \phi, \varepsilon$ which compose complex angles of the form $\varphi^c = \varphi - i\epsilon, \theta^c = \theta - i\tau, \phi^c = \phi - i\varepsilon$.

Varying $\psi(x)$ and $\overline{\psi}(x)$ independently, we obtain from (4) in accordance with (3) the following equations:

$$\Gamma_i \frac{\partial\psi(x)}{\partial x_i} + \kappa\psi(x) = 0$$
$$\qquad\qquad\qquad (i = 1, \ldots, 4) \qquad (6)$$
$$\Gamma_i^T \frac{\partial\overline{\psi}(x)}{\partial x_i} - \kappa\overline{\psi}(x) = 0.$$

Analogously, varying $\psi(\mathfrak{g})$ and $\overline{\psi}(\mathfrak{g})$ independently, one gets

$$\Gamma_k \frac{\partial\psi(\mathfrak{g})}{\partial\mathfrak{g}_k} + \kappa\psi(\mathfrak{g}) = 0$$
$$\qquad\qquad\qquad (k = 1, \ldots, 6) \qquad (7)$$
$$\Gamma_k^T \frac{\overline{\psi}(\mathfrak{g})}{\partial\mathfrak{g}_k} - \kappa\overline{\psi}(\mathfrak{g}) = 0$$

where

$$\psi(\mathfrak{g}) = \begin{pmatrix} \psi(\mathfrak{g}) \\ \dot{\psi}(\mathfrak{g}) \end{pmatrix} \qquad \Gamma_k = \begin{pmatrix} 0 & \Lambda_k^* \\ \Lambda_k & 0 \end{pmatrix}.$$

The doubling of representations, described by a bispinor $\psi(\mathfrak{g}) = (\psi(\mathfrak{g}), \dot{\psi}(\mathfrak{g}))^T$, is a well known feature of the Lorentz group representations [10, 22]. The structure of the matrices Λ_k and Λ_k^* is studied in detail in [33]. Since a universal covering $SL(2, \mathbb{C})$ of the proper orthochronous Lorentz group is a complexification of the group $SU(2)$ (see, for example, [36, 32]), it is more convenient to express the six parameters $\mathfrak{g}_k$ of the Lorentz group via the three parameters a_1, a_2, a_3 of the group $SU(2)$. It is obvious that $\mathfrak{g}_1 = a_1, \mathfrak{g}_2 = a_2, \mathfrak{g}_3 = a_3, \mathfrak{g}_4 = ia_1, \mathfrak{g}_5 = ia_2, \mathfrak{g}_6 = ia_3$. Then the first equation from (7) can be written as

$$\sum_{j=1}^{3} \Lambda_j^* \frac{\partial\dot{\psi}}{\partial\widetilde{a}_j} + i\sum_{j=1}^{3} \Lambda_j^* \frac{\partial\dot{\psi}}{\partial\widetilde{a}_j^*} + \kappa^c\psi = 0$$
$$\qquad\qquad\qquad (8)$$
$$\sum_{j=1}^{3} \Lambda_j \frac{\partial\psi}{\partial a_j} - i\sum_{j=1}^{3} \Lambda_j \frac{\partial\psi}{\partial a_j^*} + \dot{\kappa}^c\dot{\psi} = 0$$

where $a_1^* = -\mathrm{i}\mathfrak{g}_4$, $a_2^* = -\mathrm{i}\mathfrak{g}_5$, $a_3^* = -\mathrm{i}\mathfrak{g}_6$, and $\widetilde{a}_j$, $\widetilde{a}_j^*$ are the parameters corresponding to the dual basis. In essence, the equations (8) are defined in a three-dimensional complex space $\mathbb{C}^3$. In turn, the space $\mathbb{C}^3$ is isometric to a six-dimensional bivector space $\mathbb{R}^6$ (a parameter space of the Lorentz group [17, 24]). The bivector space $\mathbb{R}^6$ is a tangent space of the group manifold $\mathfrak{L}$ of the Lorentz group; that is, the manifold $\mathfrak{L}$ at each point is equivalent locally to the space $\mathbb{R}^6$. Thus, for all $\mathfrak{g} \in \mathfrak{L}$ we have $T_{\mathfrak{g}}\mathfrak{L} \simeq \mathbb{R}^6$. General solutions of the system (8) have been found in the work [33] on the tangent bundle $T\mathfrak{L} = \cup_{\mathfrak{g}\in\mathfrak{L}} T_{\mathfrak{g}}\mathfrak{L}$ of the group manifold $\mathfrak{L}$. A separation of variables in (8) is realized via the following factorization:

$$
\begin{aligned}
\psi_{lm;\dot{l}\dot{m}}^k &= \boldsymbol{f}_{lmk}^{l_0}(r)\mathfrak{M}_{l_0}^m(\varphi, \epsilon, \theta, \tau, 0, 0) \\
\psi_{\dot{l}\dot{m};lm}^{\dot{k}} &= \boldsymbol{f}_{\dot{l}\dot{m}\dot{k}}^{\dot{l}_0}(r^*)\mathfrak{M}_{\dot{l}_0}^{\dot{m}}(\varphi, \epsilon, \theta, \tau, 0, 0)
\end{aligned}
\tag{9}
$$

where $l_0 \geqslant l$, $-l_0 \leqslant m$ and $\dot{l}_0 \geqslant \dot{l}$, $-\dot{l}_0 \leqslant \dot{m}$, $\mathfrak{M}_{l_0}^m(\varphi, \epsilon, \theta, \tau, 0, 0)$ $\left(\mathfrak{M}_{\dot{l}}^{\dot{m}}(\varphi, \epsilon, \theta, \tau, 0, 0)\right)$ are associated hyperspherical functions defined on the surface of the two-dimensional complex sphere of the radius r, $\boldsymbol{f}_{lmk}^{l_0}(r)$ and $\boldsymbol{f}_{\dot{l}\dot{m}\dot{k}}^{\dot{l}_0}(r^*)$ are radial functions (for more details on the two-dimensional complex sphere, see [15, 29, 16, 33]). The associated hyperspherical function $\mathfrak{M}_l^m$ has the form

$$
\mathfrak{M}_l^m(\varphi, \epsilon, \theta, \tau, 0, 0) = \mathrm{e}^{-m(\epsilon+\mathrm{i}\varphi)} Z_m^l(\theta, \tau)
$$

where the function Z_m^l can be represented as a product of the two hypergeometric functions:

$$
Z_{mn}^l(\theta, \tau) = \cos^{2l}\frac{\theta}{2}\cosh^{2l}\frac{\tau}{2}\sum_{k=-l}^{l}\mathrm{i}^{m-k}\tan^{m-k}\frac{\theta}{2}\tanh^{-k}\frac{\tau}{2}
$$

$$
\times\,{}_2F_1\left(\begin{array}{c}m-l+1, 1-l-k \\ m-k+1\end{array}\middle|\,\mathrm{i}^2\tan^2\frac{\theta}{2}\right){}_2F_1\left(\begin{array}{c}-l+1, 1-l-k \\ -k+1\end{array}\middle|\,\tanh^2\frac{\tau}{2}\right).
$$

4. The field $(1/2, 0) \oplus (0, 1/2)$

Let us consider now an explicit construction of the relativistic wavefunction $\psi(\alpha)$ on the Poincaré group for the first nontrivial case described by the field $(1/2, 0) \oplus (0, 1/2)$ (the electron–positron field or Dirac field). In this case, the first equation from (6) coincides with the Dirac equation

$$
\mathrm{i}\gamma_n u\frac{\partial\psi(x)}{\partial x_\nu} - m\psi(x) = 0
\tag{10}
$$

where γ-matrices are defined in the standard form, that is, in the Weyl basis:

$$
\gamma_0 = \begin{pmatrix}\sigma_0 & 0 \\ 0 & -\sigma_0\end{pmatrix} \qquad \gamma_1 = \begin{pmatrix}0 & \sigma_1 \\ -\sigma_1 & 0\end{pmatrix} \qquad \gamma_2 = \begin{pmatrix}0 & \sigma_2 \\ -\sigma_2 & 0\end{pmatrix} \qquad \gamma_3 = \begin{pmatrix}0 & \sigma_3 \\ -\sigma_3 & 0\end{pmatrix}
$$

where σ_i are the Pauli matrices.

As is known, solutions of the equation (10) are found in the plane-wave approximation, that is, in the form [6, 27]

$$
\psi^+(x) = u(\mathbf{p})\,\mathrm{e}^{-\mathrm{i}px} \qquad \psi^-(x) = v(\mathbf{p})\,\mathrm{e}^{\mathrm{i}px}
$$

where the solutions $\psi^+(x)$ and $\psi^-(x)$ correspond to positive and negative energy, respectively,

and the amplitudes $u(\mathbf{p})$ and $v(\mathbf{p})$ have the following components:

$$u_1(\mathbf{p}) = \left(\frac{E+m}{2m}\right)^{1/2}\begin{bmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_+}{E+m} \end{bmatrix} \qquad u_2(\mathbf{p}) = \left(\frac{E+m}{2m}\right)^{1/2}\begin{bmatrix} 0 \\ 1 \\ \frac{p_-}{E+m} \\ \frac{-p_z}{E+m} \end{bmatrix}$$

$$v_1(\mathbf{p}) = \left(\frac{E+m}{2m}\right)^{1/2}\begin{bmatrix} \frac{p_z}{E+m} \\ \frac{p_+}{E+m} \\ 1 \\ 0 \end{bmatrix} \qquad v_2(\mathbf{p}) = \left(\frac{E+m}{2m}\right)^{1/2}\begin{bmatrix} \frac{p_-}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{bmatrix}$$

where $p_\pm = p_x \pm i p_y$.

Let us consider now solutions of the system (8) for the spin $l = 1/2$, that is, when the field is defined by a P-invariant direct sum $(1/2, 0) \oplus (0, 1/2)$. In this case the matrices Λ_i and Λ_i^* have the form

$$\Lambda_1 = \frac{1}{2}c_{\frac{1}{2}\frac{1}{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \Lambda_2 = \frac{1}{2}c_{\frac{1}{2}\frac{1}{2}}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \Lambda_3 = \frac{1}{2}c_{\frac{1}{2}\frac{1}{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\Lambda_1^* = \frac{1}{2}\dot{c}_{\frac{1}{2}\frac{1}{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \Lambda_2^* = \frac{1}{2}\dot{c}_{\frac{1}{2}\frac{1}{2}}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \Lambda_3^* = \frac{1}{2}\dot{c}_{\frac{1}{2}\frac{1}{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

It is easy to see that these matrices coincide with the Pauli matrices σ_i when $c_{\frac{1}{2}\frac{1}{2}} = 2$. The system (8) at $l = 1/2$ and $c_{\frac{1}{2},\frac{1}{2}} = \dot{c}_{\frac{1}{2},\frac{1}{2}}$ takes the form

$$-\frac{1}{2}\frac{\partial\dot{\psi}_2}{\partial\tilde{a}_1} + \frac{i}{2}\frac{\partial\dot{\psi}_2}{\partial\tilde{a}_2} - \frac{1}{2}\frac{\partial\dot{\psi}_1}{\partial\tilde{a}_3} - \frac{i}{2}\frac{\partial\dot{\psi}_2}{\partial\tilde{a}_1^*} - \frac{1}{2}\frac{\partial\dot{\psi}_2}{\partial\tilde{a}_2^*} - \frac{i}{2}\frac{\partial\dot{\psi}_1}{\partial\tilde{a}_3^*} - \kappa^c\psi_1 = 0$$

$$-\frac{1}{2}\frac{\partial\dot{\psi}_1}{\partial\tilde{a}_1} - \frac{i}{2}\frac{\partial\dot{\psi}_1}{\partial\tilde{a}_2} + \frac{1}{2}\frac{\partial\dot{\psi}_2}{\partial\tilde{a}_3} - \frac{i}{2}\frac{\partial\dot{\psi}_1}{\partial\tilde{a}_1^*} + \frac{1}{2}\frac{\partial\dot{\psi}_1}{\partial\tilde{a}_2^*} + \frac{i}{2}\frac{\partial\dot{\psi}_2}{\partial\tilde{a}_3^*} - \kappa^c\psi_2 = 0$$

$$\frac{1}{2}\frac{\partial\psi_2}{\partial a_1} - \frac{i}{2}\frac{\partial\psi_2}{\partial a_2} + \frac{1}{2}\frac{\partial\psi_1}{\partial a_3} - \frac{i}{2}\frac{\partial\psi_2}{\partial a_1^*} - \frac{1}{2}\frac{\partial\psi_2}{\partial a_2^*} - \frac{i}{2}\frac{\partial\psi_1}{\partial a_3^*} - \dot{\kappa}^c\dot{\psi}_1 = 0 \qquad (11)$$

$$\frac{1}{2}\frac{\partial\psi_1}{\partial a_1} + \frac{i}{2}\frac{\partial\psi_1}{\partial a_2} - \frac{1}{2}\frac{\partial\psi_2}{\partial a_3} - \frac{i}{2}\frac{\partial\psi_1}{\partial a_1^*} + \frac{1}{2}\frac{\partial\psi_1}{\partial a_2^*} + \frac{i}{2}\frac{\partial\psi_2}{\partial a_3^*} - \dot{\kappa}^c\dot{\psi}_2 = 0.$$

This system is defined on the tangent bundle $T\mathfrak{L}$ of the group manifold $\mathfrak{L}$. Let us find solutions of the system (11) in terms of the functions on the Lorentz group:

$$\psi_1(\mathfrak{g}) = f^l_{\frac{1}{2},\frac{1}{2}}(r)\mathfrak{M}_l^{\frac{1}{2}}(\varphi, \epsilon, \theta, \tau, 0, 0)$$

$$\psi_2(\mathfrak{g}) = f^l_{\frac{1}{2},-\frac{1}{2}}(r)\mathfrak{M}_l^{-\frac{1}{2}}(\varphi, \epsilon, \theta, \tau, 0, 0)$$

$$\dot{\psi}_1(\mathfrak{g}) = f^{\dot{l}}_{\frac{1}{2},\frac{1}{2}}(r^*)\mathfrak{M}_{\dot{l}}^{\frac{1}{2}}(\varphi, \epsilon, \theta, \tau, 0, 0)$$

$$\dot{\psi}_2(\mathfrak{g}) = f^{\dot{l}}_{\frac{1}{2},-\frac{1}{2}}(r^*)\mathfrak{M}_{\dot{l}}^{-\frac{1}{2}}(\varphi, \epsilon, \theta, \tau, 0, 0)$$

Substituting these functions into (11) and separating the variables with the aid of recurrence relations between hyperspherical functions, we come to the following system of ordinary

differential equations:

$$-\frac{1}{2}\frac{\mathrm{d}f^{\dot{l}}_{\frac{1}{2},\frac{1}{2}}(r^*)}{\mathrm{d}r^*} + \frac{1}{4r^*}f^{\dot{l}}_{\frac{1}{2},\frac{1}{2}}(r^*) + \frac{\dot{l}+\frac{1}{2}}{2r^*}f^{\dot{l}}_{\frac{1}{2},-\frac{1}{2}}(r^*) - \kappa^c f^{l}_{\frac{1}{2},\frac{1}{2}}(r) = 0$$

$$\frac{1}{2}\frac{\mathrm{d}f^{\dot{l}}_{\frac{1}{2},-\frac{1}{2}}(r^*)}{\mathrm{d}r^*} - \frac{1}{4r^*}f^{\dot{l}}_{\frac{1}{2},-\frac{1}{2}}(r^*) - \frac{\dot{l}+\frac{1}{2}}{2r^*}f^{\dot{l}}_{\frac{1}{2},\frac{1}{2}}(r^*) - \kappa^c f^{l}_{\frac{1}{2},-\frac{1}{2}}(r) = 0$$

$$\frac{1}{2}\frac{\mathrm{d}f^{l}_{\frac{1}{2},\frac{1}{2}}(r)}{\mathrm{d}r} - \frac{1}{4r}f^{l}_{\frac{1}{2},\frac{1}{2}}(r) - \frac{l+\frac{1}{2}}{2r}f^{l}_{\frac{1}{2},-\frac{1}{2}} - \dot{\kappa}^c f^{\dot{l}}_{\frac{1}{2},\frac{1}{2}}(r^*) = 0$$

$$-\frac{1}{2}\frac{\mathrm{d}f^{l}_{\frac{1}{2},-\frac{1}{2}}(r)}{\mathrm{d}r} + \frac{1}{4r}f^{l}_{\frac{1}{2},-\frac{1}{2}}(r) + \frac{l+\frac{1}{2}}{2r}f^{l}_{\frac{1}{2},\frac{1}{2}} - \dot{\kappa}^c f^{\dot{l}}_{\frac{1}{2},-\frac{1}{2}}(r^*) = 0.$$

For brevity of exposition we suppose $f_1 = f^{l}_{\frac{1}{2},\frac{1}{2}}(r)$, $f_2 = f^{l}_{\frac{1}{2},-\frac{1}{2}}(r)$, $f_3 = f^{\dot{l}}_{\frac{1}{2},\frac{1}{2}}(r^*)$, $f_4 = f^{\dot{l}}_{\frac{1}{2},-\frac{1}{2}}(r^*)$. Then

$$-2\frac{\mathrm{d}f_3}{\mathrm{d}r^*} + \frac{1}{r^*}f_3 + \frac{2\left(\dot{l}+\frac{1}{2}\right)}{r^*}f_4 - 4\kappa^c f_1 = 0$$

$$2\frac{\mathrm{d}f_4}{\mathrm{d}r^*} - \frac{1}{r^*}f_4 - \frac{2\left(\dot{l}+\frac{1}{2}\right)}{r^*}f_3 - 4\kappa^c f_2 = 0$$

$$2\frac{\mathrm{d}f_1}{\mathrm{d}r} - \frac{1}{r}f_1 - \frac{2\left(l+\frac{1}{2}\right)}{r}f_2 - 4\dot{\kappa}^c f_3 = 0$$

$$-2\frac{\mathrm{d}f_2}{\mathrm{d}r} + \frac{1}{r}f_2 + \frac{2\left(l+\frac{1}{2}\right)}{r}f_1 - 4\dot{\kappa}^c f_4 = 0.$$

Let us assume that $f_3 = \mp f_4$ and $f_2 = \pm f_1$; then the first equation coincides with the second, and the third equations coincides with the fourth. Therefore,

$$\frac{\mathrm{d}f_4}{\mathrm{d}r^*} + \frac{\dot{l}}{r^*}f_4 - 2\kappa^c f_1 = 0$$

$$\frac{\mathrm{d}f_1}{\mathrm{d}r} - \frac{l+1}{r}f_1 + 2\dot{\kappa}^c f_4 = 0.$$

Let us consider a real part $\mathrm{Re}\,r$ of the radius of a complex sphere. It is obvious that $\mathrm{Re}\,r = \mathrm{Re}\,r^*$. Writing $z = \mathrm{Re}\,r = \mathrm{Re}\,r^*$ and excluding the function f_4 at $l = \dot{l}$, we come to the following differential equation:

$$z^2\frac{\mathrm{d}^2 f_1}{\mathrm{d}z^2} - z\frac{\mathrm{d}f_1}{\mathrm{d}z} - (l^2 - 1 - 4\kappa^c\dot{\kappa}^c z^2)f_1 = 0. \tag{12}$$

The latter equation is solvable in the Bessel functions of half-integer order:

$$f_1(z) = C_1\sqrt{\kappa^c\dot{\kappa}^c}\,zJ_l\left(\sqrt{\kappa^c\dot{\kappa}^c}\,z\right) + C_2\sqrt{\kappa^c\dot{\kappa}^c}\,zJ_{-l}\left(\sqrt{\kappa^c\dot{\kappa}^c}\,z\right).$$

Further, using recurrence relations between Bessel functions, we find

$$f_4(z) = \frac{1}{2\kappa^c}\left(\frac{l+1}{z}f_1(z) - \frac{\mathrm{d}f_1}{\mathrm{d}z}\right)$$

$$= \frac{C_1}{2}\sqrt{\frac{\dot{\kappa}^c}{\kappa^c}}\,zJ_{l+1}\left(\sqrt{\kappa^c\dot{\kappa}^c}\,z\right) - \frac{C_2}{2}\sqrt{\frac{\dot{\kappa}^c}{\kappa^c}}\,zJ_{-l-1}\left(\sqrt{\kappa^c\dot{\kappa}^c}\,z\right).$$

Therefore,

$$f^{l}_{\frac{1}{2},\frac{1}{2}}(\mathrm{Re}\,r) = C_1\sqrt{\kappa^c\dot{\kappa}^c}\,\mathrm{Re}\,r\,J_l\left(\sqrt{\kappa^c\dot{\kappa}^c}\,\mathrm{Re}\,r\right) + C_2\sqrt{\kappa^c\dot{\kappa}^c}\,\mathrm{Re}\,r\,J_{-l}\left(\sqrt{\kappa^c\dot{\kappa}^c}\,\mathrm{Re}\,r\right)$$

$$f^{\dot{l}}_{\frac{1}{2},-\frac{1}{2}}(\mathrm{Re}\,r^*) = \frac{C_1}{2}\sqrt{\frac{\dot{\kappa}^c}{\kappa^c}}\,\mathrm{Re}\,r^*\,J_{l+1}\left(\sqrt{\kappa^c\dot{\kappa}^c}\,\mathrm{Re}\,r^*\right) - \frac{C_2}{2}\sqrt{\frac{\dot{\kappa}^c}{\kappa^c}}\,\mathrm{Re}\,r^*\,J_{-l-1}\left(\sqrt{\kappa^c\dot{\kappa}^c}\,\mathrm{Re}\,r^*\right).$$

In this way, solutions of the system (11) are defined by the following functions:

$$\psi_1(r, \varphi^c, \theta^c) = \boldsymbol{f}^l_{\frac{1}{2}, \frac{1}{2}}(\operatorname{Re} r)\mathfrak{M}^{\frac{1}{2}}_l(\varphi, \epsilon, \theta, \tau, 0, 0)$$

$$\psi_2(r, \varphi^c, \theta^c) = \pm\boldsymbol{f}^l_{\frac{1}{2}, \frac{1}{2}}(\operatorname{Re} r)\mathfrak{M}^{-\frac{1}{2}}_l(\varphi, \epsilon, \theta, \tau, 0, 0)$$

$$\dot{\psi}_1(r^*, \dot{\varphi}^c, \dot{\theta}^c) = \mp\boldsymbol{f}^{\dot{l}}_{\frac{1}{2}, -\frac{1}{2}}(\operatorname{Re} r^*)\mathfrak{M}^{\frac{1}{2}}_{\dot{l}}(\varphi, \epsilon, \theta, \tau, 0, 0)$$

$$\dot{\psi}_2(r^*, \dot{\varphi}^c, \dot{\theta}^c) = \boldsymbol{f}^{\dot{l}}_{\frac{1}{2}, -\frac{1}{2}}(\operatorname{Re} r^*)\mathfrak{M}^{-\frac{1}{2}}_{\dot{l}}(\varphi, \epsilon, \theta, \tau, 0, 0)$$

where

$$l = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots;$$

$$\dot{l} = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots;$$

$$\mathfrak{M}^{\pm\frac{1}{2}}_l(\varphi, \epsilon, \theta, \tau, 0, 0) = \mathrm{e}^{\mp\frac{1}{2}(\epsilon+\mathrm{i}\varphi)}Z^{\pm\frac{1}{2}}_l(\theta, \tau)$$

$$Z^{\pm\frac{1}{2}}_l(\theta, \tau) = \cos^{2l}\frac{\theta}{2}\cosh^{2l}\frac{\tau}{2}\sum_{k=-l}^{l}\mathrm{i}^{\pm\frac{1}{2}-k}\tan^{\pm\frac{1}{2}-k}\frac{\theta}{2}\tanh^{-k}\frac{\tau}{2}$$

$$\times\,{}_2F_1\left(\begin{matrix}\pm\frac{1}{2}-l+1, 1-l-k\\ \pm\frac{1}{2}-k+1\end{matrix}\,\middle|\,\mathrm{i}^2\tan^2\frac{\theta}{2}\right){}_2F_1\left(\begin{matrix}-l+1, 1-l-k\\ -k+1\end{matrix}\,\middle|\,\tanh^2\frac{\tau}{2}\right)$$

$$\mathfrak{M}^{\pm\frac{1}{2}}_{\dot{l}}(\varphi, \epsilon, \theta, \tau, 0, 0) = \mathrm{e}^{\mp\frac{1}{2}(\epsilon-\mathrm{i}\varphi)}Z^{\pm\frac{1}{2}}_{\dot{l}}(\theta, \tau)$$

$$Z^{\pm\frac{1}{2}}_{\dot{l}}(\theta, \tau) = \cos^{2\dot{l}}\frac{\theta}{2}\cosh^{2\dot{l}}\frac{\tau}{2}\sum_{k=-\dot{l}}^{\dot{l}}\mathrm{i}^{\pm\frac{1}{2}-k}\tan^{\pm\frac{1}{2}-k}\frac{\theta}{2}\tanh^{-k}\frac{\tau}{2}$$

$$\times\,{}_2F_1\left(\begin{matrix}\pm\frac{1}{2}-\dot{l}+1, 1-\dot{l}-\dot{k}\\ \pm\frac{1}{2}-\dot{k}+1\end{matrix}\,\middle|\,\mathrm{i}^2\tan^2\frac{\theta}{2}\right){}_2F_1\left(\begin{matrix}-\dot{l}+1, 1-\dot{l}-k\\ -\dot{k}+1\end{matrix}\,\middle|\,\tanh^2\frac{\tau}{2}\right).$$

Therefore, in accordance with the factorization (5), an explicit form of the relativistic wavefunction $\psi(\alpha) = \psi(x)\psi(\mathfrak{g})$ on the Poincaré group in the case of the $(1/2, 0) \oplus (0, 1/2)$ representation is given by the following expressions:

$$\psi_1(\alpha) = \psi^+_1(x)\psi_1(\mathfrak{g}) = u_1(\mathbf{p})\,\mathrm{e}^{-\mathrm{i}px}\boldsymbol{f}^l_{\frac{1}{2}, \frac{1}{2}}(\operatorname{Re} r)\mathfrak{M}^{\frac{1}{2}}_l(\varphi, \epsilon, \theta, \tau, 0, 0)$$

$$\psi_2(\alpha) = \psi^+_2(x)\psi_2(\mathfrak{g}) = \pm u_2(\mathbf{p})\,\mathrm{e}^{-\mathrm{i}px}\boldsymbol{f}^l_{\frac{1}{2}, \frac{1}{2}}(\operatorname{Re} r)\mathfrak{M}^{-\frac{1}{2}}_l(\varphi, \epsilon, \theta, \tau, 0, 0)$$

$$\dot{\psi}_1(\alpha) = \psi^-_1(x)\dot{\psi}_1(\mathfrak{g}) = \mp v_1(\mathbf{p})\,\mathrm{e}^{\mathrm{i}px}\boldsymbol{f}^{\dot{l}}_{\frac{1}{2}, -\frac{1}{2}}(\operatorname{Re} r^*)\mathfrak{M}^{\frac{1}{2}}_{\dot{l}}(\varphi, \epsilon, \theta, \tau, 0, 0) \qquad (13)$$

$$\dot{\psi}_2(\alpha) = \psi^-_2(x)\dot{\psi}_2(\mathfrak{g}) = v_2(\mathbf{p})\,\mathrm{e}^{\mathrm{i}px}\boldsymbol{f}^{\dot{l}}_{\frac{1}{2}, -\frac{1}{2}}(\operatorname{Re} r^*)\mathfrak{M}^{-\frac{1}{2}}_{\dot{l}}(\varphi, \epsilon, \theta, \tau, 0, 0)$$

The quantities (13) form *a bispinor on the Poincaré group*, $\psi(\alpha) = (\psi_1(\alpha), \psi_2(\alpha), \dot{\psi}_1(\alpha), \dot{\psi}_2(\alpha))^T$.

It is obvious that, solving the equations for $\psi(x)$ and $\psi(\mathfrak{g})$ separately, we can find in like manner all the parametrized forms of the functions (2) for any spin. The important case of the field $(1, 0) \oplus (0, 1)$ will be considered in a separate paper.

In conclusion, we should note the following circumstance. As is known, in the standard QFT, solutions of relativistic wave equations are found in the plane-wave approximation (it is hardly too much to say that these solutions are strongly degenerate) and field operators are defined in the form of Fourier expansions (or Fourier integrals) in such solutions. In passing to

the functions on the Poincaré group and their parametrized forms, the Fourier expansions are replaced by more general transformations; that is, we come to an expansion of $\psi(\alpha)$ on the group $\mathcal{P}$ or harmonic analysis of the functions on the groups (see, for example, [25, 13, 14, 30, 28]). In this way, usual Fourier analysis of the standard QFT is replaced by harmonic analysis in the case of QFTPG. The field operators on the functions (5) and their parametrizations of the form (13) will be studied in a future work in terms of harmonic analysis on the Poincaré group.

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